

Boolean-type Retractable State-finite Automata Without Outputs¹

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Abstract

An automaton \mathbf{A} is called a retractable automaton if, for every subautomaton \mathbf{B} of \mathbf{A} , there is at least one homomorphism of \mathbf{A} onto \mathbf{B} which leaves the elements of B fixed (such homomorphism is called a retract homomorphism of \mathbf{A} onto \mathbf{B}). We say that a retractable automaton $\mathbf{A}=(A,X,\delta)$ is Boolean-type if there exists a family $\{\lambda_B \mid \mathbf{B} \text{ is a subautomaton of } \mathbf{A}\}$ of retract homomorphisms λ_B of \mathbf{A} such that, for arbitrary subautomata \mathbf{B}_1 and \mathbf{B}_2 of \mathbf{A} , the condition $B_1 \subseteq B_2$ implies $\text{Ker}\lambda_{B_2} \subseteq \text{Ker}\lambda_{B_1}$. In this paper we describe the Boolean-type retractable state-finite automata without outputs.

1 Introduction and motivation

Let $\mathbf{A} = (A, X, \delta)$ be an automaton without outputs. A subautomaton \mathbf{B} of \mathbf{A} is called a retract subautomaton if there is a homomorphism of \mathbf{A} onto \mathbf{B} which leaves the elements of B fixed. A homomorphism with this property is called a retract homomorphism of \mathbf{A} onto \mathbf{B} .

In [5], A. Nagy introduced the notion of the retractable automaton. An automaton \mathbf{A} (without outputs) is called a retractable automaton if every subautomaton of \mathbf{A} is a retract subautomaton. He proved (in Theorem 3 of [5]) that if the lattice $\mathcal{L}(\mathbf{A})$ of all congruences of an automaton \mathbf{A} is complemented then \mathbf{A} is a retractable automaton. He also defined the notion of the Boolean-type retractable automaton. We say that a retractable automaton $\mathbf{A}=(A,X,\delta)$ is Boolean-type if there exists a family $\{\lambda_B \mid \mathbf{B} \text{ is a subautomaton of } \mathbf{A}\}$ of retract homomorphisms λ_B of \mathbf{A} such that, for arbitrary subautomata \mathbf{B}_1 and \mathbf{B}_2 of \mathbf{A} , the condition $B_1 \subseteq B_2$ implies $\text{Ker}\lambda_{B_2} \subseteq \text{Ker}\lambda_{B_1}$. He proved (in Theorem 5 of [5]) that if the lattice $\mathcal{L}(\mathbf{A})$ of all congruences of an automaton \mathbf{A} is a Boolean algebra then \mathbf{A} is a Boolean-type retractable automaton.

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In [5], A. Nagy investigated the not necessarily state-finite Boolean-type retractable automata containing traps (a state c is called a trap of an automaton $\mathbf{A}=(A,X,\delta)$ if $\delta(c,x) = c$ for every $x \in X$). He proved that every Boolean-type retractable automaton containing traps has a homomorphic image which is a Boolean-type retractable automaton containing exactly one trap. Moreover, he gave a complete description of Boolean-type retractable automata containing exactly one trap.

In [2], the authors defined the notion of the strongly retract extension of automata. They proved that every state finite Boolean-type retractable automaton without outputs is a direct sum of Boolean-type retractable automata whose principal factors form a tree. Moreover, a state-finite automaton \mathbf{A} is a Boolean-type retractable automaton whose principal factors form a tree if and only if it is a strongly retract extension of a strongly connected subautomaton of \mathbf{A} by a Boolean-type retractable automaton containing exactly one trap (which is described in [5]).

In [5] and [2], some theorem gives only necessary conditions for special retractable or Boolean-type retractable state-finite automata without outputs. Paper [6] is the first one which gives a complete description of state-finite retractable automata without outputs. Using the results of [6], we give a complete description of Boolean-type retractable state-finite automata without outputs.

2 Basic notations

By an automaton without outputs we mean a system (A, X, δ) where A and X are non-empty sets, and δ maps from the Cartesian product $A \times X$ to A . We will refer to A , X and δ as the state set, the input set and the transition function of \mathbf{A} , respectively. An automaton \mathbf{A} is said to be state-finite, if the set A is finite. In this paper by an automaton we always mean a state-finite automaton without outputs. We will follow the definitions and notations of [6].

An automaton $\mathbf{B}=(B,X,\delta_B)$ is called a subautomaton of an automaton $\mathbf{A} = (A, X, \delta)$ if B is a subset of A and δ_B is the restriction of δ to $B \times X$. A subautomaton \mathbf{B} of an automaton \mathbf{A} contained by every subautomaton of \mathbf{A} is called the kernel of \mathbf{A} .

By a homomorphism of an automaton (A, X, δ) into an automaton (B, X, γ) we mean a map ϕ of the set A into the set B such that $\phi(\delta(a, x)) = \gamma(\phi(a), x)$ for all $a \in A$ and $x \in X$.

A congruence of an automaton (A, X, δ) is an equivalence α of the set A such

that, for all $a, b \in A$ and $x \in X$, the assumption $(a, b) \in \alpha$ implies $(\delta(a, x), \delta(b, x)) \in \alpha$. A congruence class α containing $a \in A$ will be denoted by $[a]_\alpha$. The kernel of a homomorphism $\phi : (A, X, \delta) \mapsto (B, X, \gamma)$, which is denoted by $\text{Ker}\phi$, is defined as the following relation of A : $\text{Ker}\phi := \{(a, b) \in A \times A : \phi(a) = \phi(b)\}$. It is clear that $\text{Ker}\phi$ is a congruence on A .

We will denote the lattice of all congruences of an automaton \mathbf{A} by $\mathcal{L}(A)$. For every $\alpha, \beta \in \mathcal{L}(A)$, $\alpha \wedge \beta := \alpha \cap \beta$ and $\alpha \vee \beta = (\alpha \cup \beta)^T$ where

$$(\alpha \cup \beta)^T = (\alpha \cup \beta) \cup ((\alpha \cup \beta) \circ (\alpha \cup \beta)) \cup \dots$$

is the transitive closure of $\alpha \cup \beta$ (here \circ denotes the usual operation on the semi-group of all binary relations on A (see [3])).

Let $\mathbf{B}=(B, X, \delta_B)$ be a subautomaton of an automaton $\mathbf{A}=(A, X, \delta)$. The relation $\varrho_B = \{(b_1, b_2) \in A \times A : b_1 = b_2 \text{ or } b_1, b_2 \in B\}$ is a congruence on \mathbf{A} . This congruence is called the Rees congruence on \mathbf{A} defined by \mathbf{B} . The ϱ_B -classes of A are B itself and every one-element set $\{a\}$ with $a \in A \setminus B$.

3 Retractable automata

Definition 1 A subautomaton \mathbf{B} of an automaton $\mathbf{A}=(A, X, \delta)$ is called a retract subautomaton if there exist a homomorphism λ_B of \mathbf{A} onto \mathbf{B} which leaves the elements of B fixed. An automaton is called retractable if its every subautomaton is retract. [5]

Theorem 1 A Rees-congruence ϱ_B defined by a subautomaton $\mathbf{B}=(B, X, \delta_B)$ of an automaton $\mathbf{A}=(A, X, \delta)$ has a complement in the lattice $(\mathcal{L}(A), \vee, \wedge)$ if and only if \mathbf{B} is a retract subautomaton.

Proof Let $\mathbf{A}=(A, X, \delta)$ be an automaton. Assume that \mathbf{B} is a subautomaton of \mathbf{A} such that the Rees congruence ϱ_B has a complement in $\mathcal{L}(A)$. By the proof of Theorem 3 of [5], \mathbf{B} is a retract subautomaton of \mathbf{A} . Conversely, assume that \mathbf{B} is a retract subautomaton of \mathbf{A} . We will show that the kernel of a retract homomorphism of \mathbf{A} onto \mathbf{B} is the a complement of the Rees congruence ϱ_B defined by \mathbf{B} . We show this by proving that, for every states $a \neq b$ of \mathbf{A} , we have $(a, b) \notin \eta_B \wedge \varrho_B$ and $(a, b) \in \eta_B \vee \varrho_B$ (here λ_B denotes the corresponding retract homomorphism of \mathbf{A} onto \mathbf{B} and $\eta_B := \text{Ker}\lambda_B$). Let a, b be arbitrary elements in A with the condition $a \neq b$.

- Case $a, b \in B$.

Then $(a, b) \notin \eta_B \Rightarrow (a, b) \notin \eta_B \cap \varrho_B = \eta_B \wedge \varrho_B$.

Furthermore $a\varrho_B b \Rightarrow (a, b) \in \varrho_B \cup \eta_B \subseteq \varrho_B \vee \eta_B$.

- Case $a \in A \setminus B, b \in B$.

In this case, it follows that $(a, b) \notin \varrho_B$ thus $(a, b) \notin \eta_B \cap \varrho_B = \eta_B \wedge \varrho_B$.

Now assume that $\lambda_B(a) = \lambda_B(b)$. In this case $(a, b) \in \eta_B$ is true by definition which implies $(a, b) \in \eta_B \cup \varrho_B \subseteq \eta_B \vee \varrho_B$.

Otherwise: $\lambda_B(a) \neq \lambda_B(b) \Rightarrow \exists c \in B : \lambda_B(a) = \lambda_B(c)$ because λ_B maps onto every element B . Thus $(a, c) \in \eta_B$ and $(c, b) \in \varrho_B$, this implies $(a, b) \in (\varrho_B \cup \eta_B)^T = \varrho_B \vee \eta_B$ by definition.

- Case $a, b \in A \setminus B$.

$(a, b) \notin \varrho_B \Rightarrow (a, b) \notin \varrho_B \cap \eta_B = \varrho_B \wedge \eta_B$. Since λ_B maps A onto B , thus exists such c and d elements of B that $\lambda_B(a) = \lambda_B(c)$ and $\lambda_B(b) = \lambda_B(d)$ holds. From $(a, c) \in \eta_B, (c, d) \in \varrho_B, (b, d) \in \eta_B$ follows $(a, b) \in (\varrho_B \cup \eta_B)^T = \varrho_B \vee \eta_B$. \square

4 Boolean-type retractable automata

Definition 2 We say that a retractable automaton $\mathbf{A} = (A, X, \delta)$ is Boolean-type if there exists a family $\{\lambda_B \mid \mathbf{B} \text{ is a subautomaton of } \mathbf{A}\}$ of retract homomorphism λ_B of \mathbf{A} such that, for arbitrary \mathbf{B}_1 and \mathbf{B}_2 subautomata of \mathbf{A} , the condition $B_1 \subseteq B_2$ implies $\text{Ker} \lambda_{B_2} \subseteq \text{Ker} \lambda_{B_1}$.

In the next, if we suppose that \mathbf{A} is a Boolean-type retractable automaton and \mathbf{C} is a subautomaton of \mathbf{A} , then λ_C will denote the retract homomorphism of \mathbf{A} onto \mathbf{C} belonging to a fix family $\{\lambda_B \mid \mathbf{B} \text{ is a subautomaton of } \mathbf{A}\}$ of retract homomorphisms λ_B of \mathbf{A} satisfying the conditions of Definition 2.

In this section we shall discuss Boolean-type retractable state-finite automata without outputs. We describe these automata using the concepts and constructions of [6].

Definition 3 We say that an automaton $\mathbf{A} = (A, X, \delta)$ is a direct sum of automata $\{\mathbf{A}_i = (A_i, X, \delta_i) \mid (i \in I)\}$ (indexed with the set I) if $A_i \cap A_j = \emptyset$ for every $i, j \in I$ with $i \neq j$, and moreover $A = \bigcup_{i \in I} A_i$.

Theorem 2 ([6]) *For a state-finite automaton $\mathbf{A} = (A, X, \delta)$ the following statements are equivalent:*

- (i) \mathbf{A} is retractable.
- (ii) \mathbf{A} is the direct sum of finitely many state-finite retractable automaton, which contain kernels being isomorphic to each other. \square

The next lemma will be used in the proof of Theorem 3 several times.

Lemma 1 *If $\mathbf{D} \subseteq \mathbf{B}$ are subautomaton of a Boolean-type retractable automaton \mathbf{A} such that $\lambda_B(a) \in D$ for some $a \in A$ then $\lambda_B(a) = \lambda_D(a)$.*

Proof. Let $c = \lambda_B(a)$. As $c \in D \subseteq B$, we have $\lambda_B(c) = c$. Thus a and c are in the same $\text{Ker}\lambda_B$ -class of A . As every $\text{Ker}\lambda_B$ -class is in a $\text{Ker}\lambda_D$ -class, we have that a and c are in the same λ_D -class and so $\lambda_D(a) = \lambda_D(c)$. As $c \in D$, we have $\lambda_D(c) = c$ and so $\lambda_D(a) = \lambda_D(c) = c = \lambda_B(a)$.

Theorem 3 *For a state-finite automaton $\mathbf{A} = (A, X, \delta)$ the following statements are equivalent:*

- (i) \mathbf{A} is a Boolean-type retractable automaton.
- (ii) \mathbf{A} is the direct sum of finitely many state-finite Boolean-type retractable automata containing kernels being isomorphic to each other.

Proof (i) \mapsto (ii): Let \mathbf{A} be a Boolean-type, retractable, state-finite automaton. Since \mathbf{A} is state-finite and retractable, then by Theorem 2 \mathbf{A} is a direct sum of finitely many, state-finite, retractable automata \mathbf{A}_i ($i \in I$) containing kernels being isomorphic to each other. Let $i_0 \in I$ be an arbitrary fixed index. Let $\{\lambda_B \mid \mathbf{B} \text{ is a subautomaton of } \mathbf{A}\}$ be a family of retract homomorphisms such that $B_1 \subseteq B_2$ implies $\text{Ker}\lambda_{B_2} \subseteq \text{Ker}\lambda_{B_1}$. It is clear that A_{i_0} is a subautomaton of A . Consider those λ_C retract homomorphisms which fulfils $C \subseteq A_{i_0}$, we shall denote these with $\{\Lambda_C \mid C \subseteq A_{i_0}\}$. Since all \mathbf{C} subautomata of \mathbf{A} that has $C \subseteq A_{i_0}$ are also subautomata of \mathbf{A}_{i_0} , therefore the family $\{\Lambda_C\}$ clearly fulfils the condition $\text{Ker}\Lambda_{C_2} \subseteq \text{Ker}\Lambda_{C_1}$ for all $C_1 \subseteq C_2 \subseteq A_{i_0}$.

(ii) \mapsto (i): Assume that the automaton \mathbf{A} is a direct sum of Boolean-type retractable automata \mathbf{A}_i ($i \in I = \{1, 2, \dots, n\}$) whose kernels \mathbf{T}_i are isomorphic to each other. Let $(\cdot)_{\varphi_{i,i}}$ denote the identical mapping of T_i ($i = 1, \dots, n$). For arbitrary $i = 1, \dots, n-1$, let $(\cdot)_{\varphi_{i,i+1}}$ denote the corresponding isomorphism of T_i onto T_{i+1} . For arbitrary $i, j \in I$ with $i < j$, let $(\cdot)\Phi_{i,j} = \varphi_{i,i+1} \circ \dots \circ \varphi_{j-1,j}$. For

arbitrary $i, j \in I$ with $i > j$, let $(\cdot)\Phi_{i,j} = \varphi_{i-1,i}^{-1} \circ \cdots \circ \varphi_{i,i+1}^{-1}$. It is clear that $\Phi_{i,j}$ is an isomorphism of T_i onto T_j . Moreover, for every $i, j, k \in I$, $\Phi_{i,j} \circ \Phi_{j,k} = \Phi_{i,k}$.

Let \mathbf{B} be a subautomaton of \mathbf{A} . Let \mathcal{B} denote the set of all indexes from $1, 2, \dots, n$ which satisfy $B_i = B \cap A_i \neq \emptyset$. If $i \in \mathcal{B}$ then $T_i \subseteq B_i$. Let $i_B = \min \mathcal{B}$.

We give a retract homomorphism Λ_B of \mathbf{A} onto \mathbf{B} . If $i \in \mathcal{B}$ then let $\Lambda_B(a) = \lambda_{B_i}(a)$ for every $a \in A_i$. If $i \in I \setminus \mathcal{B}$ (that is, $B_i = \emptyset$) then let $\Lambda_B(a) = (\lambda_{T_i}(a))\Phi_{i,i_B}$. It is easy to see that Λ_B is a retract homomorphism of \mathbf{A} onto \mathbf{B} .

We show that the set $\{\Lambda_B \mid \mathbf{B} \text{ is a subautomaton of } \mathbf{A}\}$ satisfies the condition that, for every subautomaton $\mathbf{D} \subseteq \mathbf{B}$, $\text{Ker} \Lambda_B \subseteq \text{Ker} \Lambda_D$. Let $\mathbf{D} \subseteq \mathbf{B}$ be arbitrary subautomata of \mathbf{A} . We note that $\mathcal{D} \subseteq \mathcal{B}$ and $i_B \leq i_D$. Assume

$$\Lambda_B(a) = \Lambda_B(b)$$

for some $a \in A_i$ and $b \in A_j$.

Case 1: $i \in \mathcal{D}$. In this case $i_D \leq i$. We have two subcases. If $j \in \mathcal{B}$ then

$$\lambda_{B_i}(a) = \Lambda_B(a) = \Lambda_B(b) = \lambda_{B_i}(b)$$

and so $j = i$. From this it follows that

$$\lambda_{D_i}(a) = \lambda_{D_i}(b)$$

and so

$$\Lambda_D(a) = \lambda_{D_i}(a) = \lambda_{D_i}(b) = \Lambda_D(b).$$

If $j \in I \setminus \mathcal{B}$ then

$$\lambda_{B_i}(a) = \Lambda_B(a) = \Lambda_B(b) = (\lambda_{T_j}(b))\Phi_{j,i_B} \in T_{i_B} \subseteq D_{i_B}$$

and so $i = i_B \leq i_D$. This and the above $i_D \leq i$ together imply $i = i_B = i_D$. Then By Lemma 1,

$$\Lambda_D(a) = \lambda_{D_i}(a) = \lambda_{B_i}(a).$$

As

$$\Lambda_D(b) = (\lambda_{T_j}(b))\Phi_{j,i_D},$$

we have

$$\Lambda_D(a) = \lambda_{B_i}(a) = (\lambda_{T_j}(b))\Phi_{j,i_B} = (\lambda_{T_j}(b))\Phi_{j,i_D} \Lambda_D(b).$$

Case 2: $i \notin \mathcal{D}$, but $i \in \mathcal{B}$. If $j \in \mathcal{B}$ then

$$\lambda_{B_i}(a) = \Lambda_B(a) = \Lambda_B(b) = \lambda_{B_i}(b)$$

and so $j = i$. Then $\Lambda_D(a) = \Lambda_D(b)$ (see the first subcase of Case 1). If $j \notin \mathcal{B}$ then

$$\lambda_{B_i}(a) = \Lambda_B(a) = \Lambda_B(b) = (\lambda_{T_j}(b))\Phi_{j,i_B}$$

and so $i = i_B$. Thus $\lambda_{B_i}(a) \in T_i \subseteq D_i$ and so (by Lemma 1)

$$\lambda_{B_i}(a) = \lambda_{D_i}(a) = \lambda_{T_i}(a).$$

If $i_D = i_B (= i)$ then

$$\Lambda_D(a) = \lambda_{D_i}(a) = \lambda_{B_i}(a) = (\lambda_{T_j}(b))\Phi_{j,i_B} = (\lambda_{T_j}(b))\Phi_{j,i_D} = \Lambda_D(b).$$

If $i_D > i_B (= i)$ then $A_i \cap D = \emptyset$ and so

$$\Lambda_D(a) = (\lambda_{T_i}(a))\Phi_{i,i_D}$$

and

$$\Lambda_D(b) = (\lambda_{T_j}(b))\Phi_{j,i_D}.$$

As

$$\lambda_{T_i}(a) = \lambda_{B_i}(a) = (\lambda_{T_j}(b))\Phi_{j,i_B},$$

we have

$$\begin{aligned} \Lambda_D(b) &= (\lambda_{T_j}(b))\Phi_{j,i_D} = (\lambda_{T_j}(b))(\Phi_{j,i_B} \circ \Phi_{i_B,i_D}) = \\ &= ((\lambda_{T_j}(b))\Phi_{j,i_B})\Phi_{i_B,i_D} = (\lambda_{T_i}(a))\Phi_{i_B,i_D} = (\lambda_{T_i}(a))\Phi_{i,i_D} = \Lambda_D(a). \end{aligned}$$

Case 3: $i \notin \mathcal{B}$. If $j \in \mathcal{B}$ then we can prove (as in the second subcases of Case 1 and Case 2) that $\Lambda_D(a) = \Lambda_D(b)$. Consider the case when $j \notin \mathcal{B}$. Then

$$(\lambda_{T_i}(a))\Phi_{i,i_B} = \Lambda_B(a) = \Lambda_B(b) = (\lambda_{T_j}(b))\Phi_{j,i_B}.$$

Hence

$$\begin{aligned} \Lambda_D(a) &= (\lambda_{T_i}(a))\Phi_{i,i_D} = (\lambda_{T_i}(a))(\Phi_{i,i_B} \circ \Phi_{i_B,i_D}) = ((\lambda_{T_i}(a))\Phi_{i,i_B})\Phi_{i_B,i_D} = \\ &= ((\lambda_{T_j}(b))\Phi_{j,i_B})\Phi_{i_B,i_D} = (\lambda_{T_j}(b))(\Phi_{j,i_B} \circ \Phi_{i_B,i_D}) = (\lambda_{T_j}(b))\Phi_{j,i_D} = \Lambda_D(b). \end{aligned}$$

In all cases, we have that $\Lambda_B(a) = \Lambda_B(b)$ implies $\Lambda_D(a) = \Lambda_D(b)$ for every $a, b \in A$. Consequently

$$\text{Ker}\Lambda_B \subseteq \text{Ker}\Lambda_D.$$

Hence A is a Boolean-type retractable automaton. \square

By Theorem 3, we can focus our attention on a Boolean-type retractable automaton containing a kernel. In our investigation two notions will play important role. These notions are the dilation of automata and the semi-connected automata.

Definition 4 Let \mathbf{B} be an arbitrary subautomaton of an automaton $\mathbf{A} = (A, X, \delta)$. We say that \mathbf{A} is a dilation of \mathbf{B} if there exists a mapping $\phi_{dil}(\cdot)$ of A onto B that leaves the elements of B fixed, and fulfils $\delta(a, x) = \delta_B(\phi_{dil}(a), x)$ for all $a \in A$ and $x \in X$. This fact will be denoted by: $(A, X, \delta; B, \phi_{dil})$. ([5])

If a is an arbitrary element of an \mathbf{A} automaton, then let $R(a)$ denote the subautomaton generated by the element a (the smallest subautomaton containing a). It is easy to see that

$$R(a) = \{\delta(a, x) : x \in X^*\},$$

where X^* is the free monoid over X . Let us define the following relation:

$$\mathcal{R} := \{(a, b) \in A \times A : R(a) = R(b)\}.$$

It is evident that \mathcal{R} is an equivalence relation. The \mathcal{R} class containing a particular a element is denoted by R_a . The set $R(a) \setminus R_a$ is denoted by $R[a]$. It is clear that $R[a]$ is either empty set or a subautomaton of \mathbf{A} . $R\{a\} = R(a)/\rho_{R[a]}$ factor automaton is called a principal factor of \mathbf{A} . If $R[a]$ is an empty set, then consider $R\{a\}$ as $R(a)$. [6]

An \mathbf{A} automaton is said to be strongly connected if, for any $a, b \in A$ elements, there exist a word $p \in X^+$ such that $\delta(a, p) = b$; (X^+ is the free semigroup over X). Remark: for a word $p = x_1 x_2 \dots x_n$ and an element a the transition function is defined as the following:

$$\delta(a, p) = \delta(\dots \delta(\delta(a, x_1), x_2) \dots x_n).$$

An automaton is called strongly trap connected if it contains exactly one trap and, for every $a \in A \setminus \{trap\}$ and $b \in A$, there is a word $p \in X^+$ such that $\delta(a, p) = b$.

An automaton is said to be semi-connected if its every principal factor is either strongly connected or strongly trap connected. ([6])

Theorem 4 ([6]) A state-finite automaton without outputs is a retractable automaton if and only if it is a dilation of a semi-connected retractable automaton. \square

The next theorem is the extension of Theorem 4.

Theorem 5 A state-finite automaton without outputs is a Boolean-type retractable automaton if and only if it is a dilation of a semi-connected Boolean-type retractable automaton.

Proof. Let \mathbf{A} be a Boolean-type retractable state- finite automaton without outputs. Then, by Theorem 4, \mathbf{A} is a dilation of the retractable semi-connected automaton \mathbf{C} . For a subautomaton \mathbf{B} of \mathbf{C} , let λ'_B denote the restriction of λ_B to \mathbf{C} . It is easy to see that \mathbf{C} is a Boolean-type retractable automaton with the family $\{\lambda'_B \mid \mathbf{B} \text{ is a subautomaton of } \mathbf{C}\}$.

Conversely, let the automaton $\mathbf{A}=(A,X,\delta;\mathbf{B},\phi_{dil})$ be a dilation of the automaton $\mathbf{B}=(B,X,\delta_B)$. Assume that \mathbf{B} is Boolean-type retractable with the family $\{\lambda_C \mid \mathbf{C} \text{ is a subautomaton of } \mathbf{B}\}$. Since all subautomata of \mathbf{A} are subautomata of \mathbf{B} , it is clear that, for every subautomaton \mathbf{C} of \mathbf{A} , $\lambda_C \circ \phi_{dil}$ is a retract homomorphism of \mathbf{A} onto \mathbf{C} . Moreover, \mathbf{A} is a Boolean-type retractable automaton with the family $\{\lambda_C \circ \phi \mid \mathbf{C} \text{ is a subautomaton of } \mathbf{A}\}$. \square

By Theorem 5 and Theorem 3, we can concentrate our attention on semi-connected automata containing kernels.

Definition 5 Let (T, \leq) be a partially ordered set, in which every two element subset has a lower bound, and every non-empty subset of T having an upper bound contains a maximal element. Consider the operation on T which maps a couple $(t_1, t_2) \in T \times T$ to the (unique) greatest upper bound of the set $\{t_1, t_2\}$. T is a semilattice under this operation. This semilattice is called a tree. It is clear that every finite tree has a least element. ([7])

If a non-trivial state-finite automaton \mathbf{A} contains exactly one trap a_0 then A^0 will denote the set $A \setminus a_0$. If A is a trivial automaton, then let $A^0 = A$. On the set $A^0 \times X$ we consider a partial (transition) function δ^0 which is defined only on couples (a, x) for which $\delta(a, x) \in A^0$; in this case $\delta^0(a, x) = \delta(a, x)$. We shall say that (A^0, X, δ^0) is the partial automaton derived from the automaton \mathbf{A} .

If \mathbf{A}^0 and \mathbf{B}^0 are partial automata, then a mapping ϕ of A^0 into B^0 is called a partial homomorphism of \mathbf{A}^0 into \mathbf{B}^0 if, for every $a \in A^0$ and $x \in X$, the condition $\delta_A(a, x) \in A^0$ implies $\delta_B(\phi(a), x) \in B^0$ and $\delta_B(\phi(a), x) = \phi(\delta(a, x))$.

Construction ([6]) Let (T, \leq) be a finite tree with the least element i_0 . Let $i > j$ ($i, j \in T$) denote the fact that $i \geq j$ and for all $k \in T$, the condition $i \geq k \geq j$ implies $i = k$ or $j = k$. Let $\mathbf{A}_i = (A_i, X, \delta_i)$, $i \in T$ be a family of pairwise disjoint automata satisfying the following conditions:

- (i) \mathbf{A}_{i_0} is strongly connected and \mathbf{A}_i is strongly trap connected for every $i \in T, i \neq i_0$.

- (ii) Let $\phi_{i,i}$ denote the identical mapping of \mathbf{A}_i . Assume that, for every $i, j \in T, i > j$, there exist a homomorphism $\phi_{i,j}$ which maps \mathbf{A}_i^0 into \mathbf{A}_j^0 such that
- (iii) for every $i > j$ there exist elements $a \in A_i^0$ and $x \in X$ such that $\delta_i(a, x) \notin A_i^0$, $\delta_j(\phi_{i,j}(a), x) \in A_j^0$.

For arbitrary elements $i, j \in T$ with $i \geq j$, we define a partial homomorphism $\Phi_{i,j}$ of \mathbf{A}_i^0 into \mathbf{A}_j^0 as follows: $\Phi_{i,i} = \phi_{i,i}$ and, if $i > j$ such that $i > k_1 > \dots > k_n > j$, then let

$$\Phi_{i,j} = \phi_{k_n,j} \circ \phi_{k_{n-1},k_n} \circ \dots \circ \phi_{k_1,k_2} \circ \phi_{i,k_1}.$$

(We note that if $i \geq j \geq k$ are arbitrary elements of T , then $\Phi_{i,k} = \Phi_{j,k} \circ \Phi_{i,j}$.)

Let $A = \bigcup_{i \in T} A_i^0$. Define a transition function $\delta' : A \times X \mapsto A$ as follows. If $a \in A_i^0$ and $x \in X$ then let

$$\delta'(a, x) = \delta_{i'[a,x]}(\Phi_{i,i'[a,x]}(a), x),$$

where $i'[a, x]$ denotes the greatest element of the set $\{j \in T : \delta_j(\Phi_{i,j}(a), x) \in A_j^0\}$. It is clear that $\mathbf{A} = (A, X, \delta')$ is an automaton which will be denoted by $(A_i, X, \delta_i; \phi_{i,j}, T)$.

Theorem 6 ([6]) *A state-finite automaton without outputs is a semi connected retractable automaton containing a kernel if and only if it is isomorphic to an automaton $(A_i, X, \delta_i; \phi_{i,j}, T)$ defined in the Construction.* \square

Remark 1 By the proof of Theorem 7 of [6] if \mathbf{R} is a subautomaton of an automaton $(A_i, X, \delta_i; \phi_{i,j}, T)$ constructed as above, then there is an ideal $\Gamma \subseteq T$ such that $R = \bigcup_{j \in \Gamma} A_j^0$. As T is a tree

$$\pi : i \mapsto \max\{\gamma \in \Gamma : \gamma \leq i\}$$

is a well defined mapping of T onto Γ which leaves the elements of Γ fixed. λ_R defined by $\lambda_R(a) = \Phi_{i,\pi(i)}(a)$ ($a \in A_i^0$) is a retract homomorphism of \mathbf{A} onto \mathbf{R} . ([6]) This fact will be used in the proof of the next Theorem.

Theorem 7 *A state-finite automaton without outputs is a semi-connected Boolean-type retractable automaton containing a kernel if and only if it is isomorphic to an automaton $(A_i, X, \delta_i; \phi_{i,j}, T)$ defined in the Construction.*

Proof Let \mathbf{A} be a state-finite automaton without outputs which contains a kernel. Assume that \mathbf{A} is also semi-connected and Boolean-type retractable. Then, by Theorem 6, \mathbf{A} is isomorphic to an automaton $\mathbf{A}=(A_i, X, \delta_i; \phi_{i,j}, T)$ which is defined in the Construction.

The main part of the proof is to show that every automaton $\mathbf{A}=(A_i, X, \delta_i; \phi_{i,j}, T)$ constructed as above is Boolean-type retractable. According to Theorem 6 the automaton $\mathbf{A}=(A_i, X, \delta_i; \phi_{i,j}, T)$ is retractable. Let \mathbf{B} be a subautomaton of \mathbf{A} . By Remark 1 there is an ideal $\Gamma \subseteq T$ such that $B = \bigcup_{j \in \Gamma} A_j^0$. Let $\pi_B : i \mapsto \{\gamma \in \Gamma : \gamma \leq i\}$. For every $a \in A_j (j \in T)$ let $\lambda_B(a) := \Phi_{j, \pi_B(j)}(a)$. Using also Remark 1, it is easy to see that λ_B is a retract homomorphism of \mathbf{A} onto \mathbf{B} . Let B_1 and B_2 be arbitrary subautomata with $B_1 \subseteq B_2$. We will show that $\text{Ker} \lambda_{B_2} \subseteq \text{Ker} \lambda_{B_1}$. Assume $\lambda_2(a) = \lambda_2(b)$ for some $a, b \in A$. According to Remark 1, $\lambda_{B_1} = \Phi_{\pi_{B_2}(j), \pi_{B_1}(j)} \circ \Phi_{j, \pi_{B_2}(j)}$. Thus

$$\begin{aligned} \lambda_{B_1}(a) &= (\Phi_{\pi_{B_2}(i), \pi_{B_1}(i)} \circ \Phi_{j, \pi_{B_2}(i)})(a) = (\Phi_{\pi_{B_2}(i), \pi_{B_1}(i)} \circ \lambda_{B_2})(a) = \\ &= (\Phi_{\pi_{B_2}(i), \pi_{B_1}(i)} \circ \lambda_{B_2})(b) = (\Phi_{\pi_{B_2}(i), \pi_{B_1}(i)} \circ \Phi_{j, \pi_{B_2}(i)})(b) = \lambda_{B_1}(b). \end{aligned}$$

Consequently $\text{Ker} \lambda_{B_2} \subseteq \text{Ker} \lambda_{B_1}$. Hence $\mathbf{A}=(A_i, X, \delta_i; \phi_{i,j}, T)$ is a Boolean-type retractable automaton with the family $\{\lambda_B \mid \mathbf{B} \text{ is a subautomaton of } \mathbf{A}\}$. \square

References

- [1] Babcsányi I. and Nagy, A., *Retractable Mealy-automata*, Pure Mathematics and Applications, 3 (1992), 147-153
- [2] Babcsányi, I., Nagy, A., *Boolean-type retractable automata*, Publicationes Mathematicae, Debrecen, 48/3-4 (1996), 193-200
- [3] Clifford, A.H. and Preston, G. B., *The Algebraic Theory of Semigroups I.*, Amer. Math. Soc., R.I., 1961
- [4] Gécseg, F. and I. Peák, *Algebraic Theory of Automata*, Akadémia Kiadó, Budapest 1972
- [5] Nagy, A., *Boolean-type retractable automata with traps*, Acta Cybernetica, 10 (1991), 53 - 64
- [6] Nagy, A., *Retractable state-finite automata without outputs*, Acta Cybernetica 16 (2004), 399 - 409

- [7] Tully, E.J., *Semigroups in which each ideal is a retract*, J Austral Math. Soc.
9 (1969), 239-245